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Correlation Inequalities for Interacting Particle Systems with Duality

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Abstract We prove a *comparison inequality* between a system of independent random walkers and a system of random walkers which either interact by *attracting each other*—a process which we call here the symmetric inclusion process (SIP)—or *repel each other*—a generalized version of the well-known symmetric exclusion process. As an application, new *correlation inequalities* are obtained for the SIP, as well as for some interacting diffusions which are used as models of heat conduction,—the so-called Brownian momentum process, and the Brownian energy process. These inequalities are counterparts of the inequalities (in the opposite direction) for the symmetric exclusion process, showing that the SIP is a natural bosonic analogue of the symmetric exclusion process, which is fermionic. Finally, we consider a boundary driven version of the SIP for which we prove duality and then obtain correlation inequalities.

Keywords Exclusion process · Duality · Correlation inequalities · Heat conduction

1 Introduction

In Liggett [16], Chap. VIII, Proposition 1.7, a comparison inequality between independent symmetric random walkers and corresponding exclusion symmetric random walkers is obtained. This inequality plays a crucial role in the understanding of the exclusion process (SEP); it makes rigorous the intuitive picture that symmetric random walkers interacting

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by exclusion are more spread out than the corresponding independent walkers, as a consequence of their repulsive interaction (exclusion), or in more physical terms, because of the fermionic nature of the exclusion process. The comparison inequality is a key ingredient in the ergodic theory of the symmetric exclusion process, i.e., in the characterization of the invariant measures, and the measures which are in the course of time attracted to a given invariant measure. The comparison inequality has been generalized later on by Andjel [1], Liggett [15], and recently in the work of Borcea, Brändén and Liggett [5].

In the search of a natural conservative particle system where the opposite inequality holds, i.e., where the particles are *less spread out* than corresponding independent random walkers, it is natural to think of a "bosonic counterpart" of the exclusion process. In fact, such a process was introduced in [10] and [11] as the *dual* of the Brownian momentum process, a stochastic model of heat conduction (similar models of heat conduction were introduced in [3] and [9], see also [2] for the study of the structure function in a natural asymmetric version, and [14] for the study of ergodicity in the infinite particle system).

In the present paper we analyze this "bosonic counterpart" of the exclusion process. We will call this process (as will be motivated by a Poisson clock representation) the "symmetric inclusion process" (SIP). In the SIP, jumps are performed according to independent random walks, and on top of that particles "invite" other particles to join their site (inclusion). For this process we prove the analogue of the comparison inequality for the symmetric exclusion process. From the comparison inequality, using the knowledge of the stationary measure and the self-duality property of the process, we deduce a series of correlation inequalities. Again, in going from exclusion to inclusion process the correlations turn from negative to positive. We remark however that these positive correlation inequalities are different from the ordinary preservation of positive correlations for monotone processes [13], because the SIP is not a monotone process. Since the SIP is dual to the heat conduction model it is immediate to extend those correlation inequalities to the Brownian momentum process and the Brownian energy process.

We also introduce the non-equilibrium versions of the SIP, i.e., we consider the boundary driven version of SIP. In this case, for appropriate choice of the boundary generators, we prove duality of the process to a SIP model with absorbing boundary condition. We then deduce a correlation inequality, explaining and generalizing the positivity of the covariance in the non-equilibrium steady state of the heat conduction model in [10].

All the results will be stated in the context of a family of SIP(m) models, which are labeled by parameter $m \in \mathbb{N}$. As the SEP model can be generalized to the situation where there are at most $n \in \mathbb{N}$ particles per site (this corresponds to a quantum spin chain with SU(2) symmetry and spin value $j = n/2$), in the same way the SIP model can be extended to represent the situation of a quantum spin chain with $\mathrm{SU}(1,1)$ symmetry and spin value $k = m/4$ [10].

The paper is organized as follows. In Sect. 2 we define the SIP(m) process, restricting to a context where its existence can be immediately established. The main comparison inequality, which allows to compare SIP walkers to independent walkers (by a suitable generalization of Liggett comparison inequality) is proved in Sect. 3. Correlation inequalities for the SIP(m) process that can be deduced from the comparison inequality are proved in Sect. 5 (the necessary knowledge of the stationary measure and the self-duality property are presented in Sect. 4). In particular, in Sect. 5 it is proved that when the SIP(m) process is started from its stationary measure then correlations are always positive, while when the process is initialized with a general product measure then positivity of correlations is recovered in the long time limit. Further correlation inequalities for systems similar to the SIP(m) process are discussed in the subsequent sections. Repulsive interaction (the SEP(n), which generalize the standard SEP) is presented in Sect. 6. Some interacting diffusions dual to the SIP(m) process are studied in Sect. 7. Finally the boundary driven SIP(m) process is analyzed in Sect. 8.

2 Definition

In the whole of the paper, S will denote or a finite set, or $S = \mathbb{Z}^d$. Next, $p(x,y)$ denotes an irreducible (discrete-time) symmetric random walk transition probability on S, i.e., $p(x,y) = p(y,x) \geq 0$, $\sum_y p(x,y) = 1$, and $p(x,x) = 0$. In the case $S = \mathbb{Z}^d$, we suppose furthermore that $p(x,y)$ is finite range and translation invariant, i.e., $p(x,y) = \pi(y-x)$, and there exists $R > 0$ such that $p(x,y) = 0$ for $|x-y| > R$. This assumption for the infinite-volume case avoids technical problems for the existence of the SIP(m) which for the subject of this paper are irrelevant. The proof of existence of the SIP(m) in our infinite-volume context (with the process started from a "tempered" initial configuration, i.e. $\eta(y) \leq |y|^k$ for some k and for all y) follows from self-duality, along the lines of [6], Chap. 2.

The symmetric inclusion process with parameter $m \in (0,\infty)$ associated to the transition kernel p is the Markov process on $\Omega := \mathbb{N}^S$ with generator defined on the core of local functions by

$$Lf(\eta) = \sum_{x,y \in S} p(x,y) 2\eta_x (m + 2\eta_y)\big(f(\eta^{x,y}) - f(\eta)\big) \tag{2.1}$$

where, for $\eta \in \Omega$, $\eta^{x,y}$ denotes the configuration obtained from η by removing one particle from x and putting it at y.

In [10], for $m = 1$ this model was introduced as the dual of a model of heat conduction, the so-called Brownian momentum process, see also [11], and [3] for generalized and or similar models of heat conduction.

The process with generator (2.1) can be interpreted as follows. Every particle has two exponential clocks: one clock—the so-called random walk clock—has rate $2m$, the other clock—the so-called inclusion clock—has rate 4. When the random walk clock of a particle at site $x \in S$ rings, the particle performs a random walk jump with probability $p(x,y)$ to site $y \in S$. When the inclusion process clock rings at site $y \in S$, with probability $p(y,x) = p(x,y)$ a particle from site $x \in S$ is selected and joins site y.

From this interpretation, we see that besides jumps of a system of independent random walkers, this system of particles has the tendency to bring particles together at the same site (inclusion), and can therefore be thought of as a "bosonic" counterpart of the symmetric exclusion process. We shall discuss more the bosonic nature of the inclusion process versus to the fermionic nature of the exclusion process in Sect. 3, where they both will be compared to a system of a finite number of independent random walkers. For the bosonic/fermonic representation of inclusion/exclusion process via creation and annihilation operators we refer to [10].

To make the analogy with the exclusion process even more transparent, in an exclusion process with at most n particles ($n \in \mathbb{N}$) per site (notation SEP(n)), the jump rate is $\eta_i(n - \eta_j)p(i,j)$. Apart from a global factor 4, the SIP(m) is obtained by changing the minus into a plus and choosing $n = m/2$.

Notice that the rates in (2.1) are increasing both in the number of particles of the departure and in the number of particles of the arrival site (the rate is $p(x,y)2\eta_x(m + 2\eta_y)$ for a particle to jump from x to y). Therefore, by the necessary and sufficient conditions of [12], Theorem 2.21, the SIP is not a monotone process. It is also easy to see that due to the attraction between particles in the SIP, there cannot be a coupling that preserves the order of configurations, i.e., in any coupling starting from an unequal ordered pair of configurations, the order will be lost in the course of time with positive probability.

2.1 Assumptions on the Transition Probability Kernel

In this section we introduce the assumptions that we need to prove the positivity of correlations of stationary measures obtained as limits of general initial product measures (see later for precise definitions). This assumptions are only relevant in the infinite volume case $S = \mathbb{Z}^d$ and they are indeed satisfied in the context of finite-range translation-invariant underlying random walk kernel $p(x, y) = \pi(y - x)$. However, all our results on correlation inequalities for stationary measures depend only on one or both of the assumptions below, i.e., if on more general graphs, or on $\mathbb{Z}^d$ with more general $p(x, y)$, existence of SIP(m) would be established, then the corresponding correlation inequalities hold under one or both of the assumptions (A1), (A2) below.

We define the associated continuous-time random walk transition probabilities of random walk jumping at rate $2m$:

$$p_t(x, y) = \sum_{n=0}^{\infty} \frac{(2mt)^n}{n!} e^{-2mt} p^{(n)}(x, y) \tag{2.2}$$

where $p^{(n)}$ denotes the nth power of the transition matrix p. Denote by $\mathbb{P}_{x,y}^{\mathrm{IRW}(m)}$ the probability measure on path space associated to two independent random walkers X_t, Y_t started at x, y and jumping according to (2.2) and by $\mathbb{P}_{x,y}^{\mathrm{SIP}(m)}$ the corresponding probability for two SIP walkers $X_t^{'}, Y_t^{'}$ jumping with the rates of generator (2.1).

We consider two assumptions

– Assumption (A1)

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{\mathrm{IRW}(m)}(X_t = Y_t) = 0 \tag{2.3}$$

– Assumption (A2)

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{\mathrm{SIP}(m)}(X_t^{'} = Y_t^{'}) = 0 \tag{2.4}$$

The assumption (A1) amounts to requiring that for large $t > 0$, two independent random walkers walking according to the continuous time random walk probability (2.2) will be at the same place with vanishing probability. The assumption (A1) follows immediately if we have

$$\lim_{t\to\infty} \sup_{x,y} p_t(x, y) = 0 \tag{2.5}$$

since then

$$\lim_{t\to\infty} \sup_{x,y} \mathbb{P}_{x,y}^{\mathrm{IRW}(m)}(X_t = Y_t) = \lim_{t\to\infty} \sup_{x,y} \sum_{u\in S} p_t(x, u) p_t(y, u) = \lim_{t\to\infty} \sup_{x,y} p_{2t}(x, y) = 0 \tag{2.6}$$

Notice also that, by simple rescaling of time, (A1) holds for all $m > 0$ as soon as it holds for some $m > 0$.

Assumption (A2) guarantees that two walkers evolving with the SIP dynamic will be typically at different positions at large times. Notice that in the case we consider, i.e., the translation invariant finite-range case $S = \mathbb{Z}^d$, $p(x, y) = p(0, y - x) =: \pi(y - x)$, this is automatically satisfied, as the difference walk $X_t^{'} - Y_t^{'}$ of two SIP particles is a random walk Z_t on $\mathbb{Z}^d$ with generator

$$L^Z f(z) = 8\pi(z)(f(0) - f(z)) + \sum_y 4m\pi(y)(f(z + y) - f(z)) \tag{2.7}$$

which is clearly not positive recurrent.

Assumption (A2) implies that any finite number of SIP particles will eventually be at different locations. This is made precise in Lemma 1 in Sect. 5.

3 Comparison of the SIP with Independent Random Walks

We will first consider the SIP process with a finite number of particle in Sect. 3.1 and then state the comparison inequality in Sect. 3.2.

3.1 The Finite SIP

If we start the SIP with n particles at positions $x_1, \ldots, x_n \in S$, we can keep track of the labels of the particles. This gives then a continuous-time Markov chain on S^n with generator

$$\begin{aligned}\mathcal{L}_n f(x_1, \ldots, x_n) &= \sum_{i=1}^n \sum_{y \in S} 2p(x_i, y)\left(m + 2\sum_{j=1}^n I(y = x_j)\right)(f(x^{x_i,y}) - f(x)) \\ &= \mathcal{L}_{1,n} f(x) + \mathcal{L}_{2,n} f(x)\end{aligned} \tag{3.1}$$

where $x^{x_i,y}$ denotes the n-tuple $(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_n)$. Further, $\mathcal{L}_{1,n}$, resp. $\mathcal{L}_{2,n}$ denote the random walk resp. inclusion part of the generator and are defined as follows

$$\mathcal{L}_{1,n} f(x_1, \ldots, x_n) = \sum_{i=1}^n \sum_{y \in S} 2mp(x_i, y)(f(x^{x_i,y}) - f(x)) \tag{3.2}$$

$$\mathcal{L}_{2,n} f(x_1, \ldots, x_n) = \sum_{i=1}^n \sum_{j=1}^n 4p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \tag{3.3}$$

3.2 Comparison Inequality

From the description above, it is intuitively clear that in the SIP, particle tend to be less spread out than in a system of independent random walkers. Theorem 1 below formalizes this intuition and is the analogue of a comparison inequality of the SEP ([16], Chap. VIII, Proposition 1.7).

To formulate it, we need the notion of a positive definite function. A bounded function $f : S \times S \to \mathbb{R}$ is called positive definite if for all $\beta : S \to \mathbb{R}$ such that $\sum_x |\beta(x)| < \infty$

$$\sum_{x,y} f(x, y)\beta(x)\beta(y) \geq 0$$

A function $f : S^n \to \mathbb{R}$ is called positive definite if it is positive definite in every pair of variables.

We first introduce a slightly more general operator with parameters $a > 0$, $b \in \mathbb{R}$ that includes both process of exclusion and inclusion type.

$$\mathcal{L}_n^{a,b} f(x_1, \dots, x_n) = \sum_{i=1}^{n} \sum_{y \in S} p(x_i, y) \left(a + b \sum_{j=1}^{n} I(y = x_j) \right) (f(x^{x_i,y}) - f(x)) \tag{3.4}$$

so

$$\mathcal{L}_n^{a,b} = \mathcal{L}_{1,n}^{a} + \mathcal{L}_{2,n}^{b}$$

where

$$\mathcal{L}_{1,n}^{a} f(x_1, \dots, x_n) = a \sum_{i=1}^{n} \sum_{y \in S} p(x_i, y)(f(x^{x_i,y}) - f(x)) \tag{3.5}$$

is the independent random walk part (random walks jumping at rate a) and

$$\mathcal{L}_{2,n}^{b} f(x_1, \dots, x_n) = b \sum_{i=1}^{n} \sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \tag{3.6}$$

is the "clumping" part, i.e., when $b < 0$ clumping is discouraged, and $b > 0$ clumping is favored.

For values of a, b such that $\mathcal{L}_n^{a,b}$ is a generator, we call $T_n^{a,b}(t)$ the semigroup on functions $f : S^n \to \mathbb{R}$ associated to the generator (3.4), and $U_n^a(t)$ the semigroup of a system of independent continuous-time random walkers (jumping at rate a), i.e., the semigroup associated to the generator $\mathcal{L}_{1,n}^{a}$ in (3.5). Notice that when $b < 0$, $T_n^{a,b}(t)$ is not always a Markov semigroup. However, for the applications of negative b, we have in mind generalized exclusion process (see Sect. 6) in which case $a = n$ and $b = -1$ (where n denotes the maximum number of allowed particles per site), and in this case $T_n^{a,b}(t)$ is a Markov semigroup.

Theorem 1 *Let $f : S^n \to \mathbb{R}$ be positive definite, bounded and symmetric. Then we have for $b > 0$*

$$U_n^a(t) f \le T_n^{a,b}(t) f \tag{3.8}$$

and for $b < 0$, if $(T^{a,b}(t))_{t \ge 0}$ is a Markov semigroup, we have

$$U_n^a(t) f \ge T_n^{a,b}(t) f \tag{3.9}$$

Proof The proof follows the proof in [16], but for the sake of self-consistency we prefer to give it explicitely. Suppose $b > 0$.

Start with the decomposition (3.4) and use the symmetry of $p(x, y)$ and f to write

$$\begin{aligned}
(\mathcal{L}_n^{a,b} f - \mathcal{L}_{1,n}^{a} f)(x) &= (\mathcal{L}_{2,n}^{b} f)(x) \\
&= b \sum_{i=1}^{n} \sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) - f(x)) \\
&= \frac{b}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} p(x_i, x_j)(f(x^{x_i,x_j}) + f(x^{x_j,x_i}) - 2f(x))
\end{aligned}$$

$$
\begin{aligned}
&= \frac{b}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} p(x_i, x_j) \\
&\quad \times \sum_{u,v} f(x_1, \ldots, x_{i-1}, u, x_{i+1}, \ldots, x_{j-1}, v, x_{j+1}, \ldots, x_n) \\
&\quad \times (\delta_{x_i,u} - \delta_{x_j,u})(\delta_{x_i,v} - \delta_{x_j,v}) \\
&\geq 0
\end{aligned} \tag{3.10}
$$

where in the last step we used that f is positive definite.

Since $U_n^a(t)$ is the semigroup of independent walks, it maps positive definite functions into positive definite functions, and so we have

$$
\big(\mathcal{L}_n U_n^a(t) f - \mathcal{L}_{1,n}^a U_n^a(t) f\big) = \mathcal{L}_{2,n}^b U_n^a(t) f \geq 0
$$

We can then use the variation of constants formula

$$
T_n^{a,b}(t) f - U_n^a(t) f = \int_0^t ds\; T_n^{a,b}(t-s)\big(\mathcal{L}_{2,n}^b U_n^a(s) f\big) \geq 0 \tag{3.11}
$$

and remember that $T_n^{a,b}(t)$ is a Markov semigroup which therefore maps non-negative functions into non-negative functions.

The proof for $b < 0$, under the assumption that $T_n^{a,b}(t)$ is a Markov semigroup is identical. □

4 Stationary Measures and Self-duality for the SIP(m)

The stationary and reversible measures of SIP(m) are product measures of "discrete gamma distributions"

$$
\nu_\lambda(d\eta) = \bigotimes_{x \in S} \nu_\lambda^m(d\eta_x)
$$

where for $n \geq 0$

$$
\nu_\lambda^m(n) = \frac{1}{Z_{\lambda,m}} \frac{\lambda^n}{n!} \frac{\Gamma(\frac{m}{2} + n)}{\Gamma(\frac{m}{2})}, \quad n \in \mathbb{N} \tag{4.1}
$$

with $0 \leq \lambda < 1$ a parameter, $\Gamma(r)$ the gamma-function and

$$
Z_{\lambda,m} = \left(\frac{1}{1-\lambda}\right)^{m/2}
$$

Notice that for $m = 2$, ν_λ^m is a geometric distribution (starting from zero), i.e., $\nu_\lambda^2(n) = \lambda^n(1-\lambda), n \in \mathbb{N}$ and for $m/2$ an integer ν_λ^m is negative binomial distribution $NB(m/2, \lambda)$. Moreover, the measures ν^m have the following convolution property

$$
\nu_\lambda^m * \nu_\lambda^l = \nu_\lambda^{m+l} \tag{4.2}
$$

where $*$ denotes convolution, i.e., a sample from $\nu_\lambda^m * \nu_\lambda^l$ is obtained by site-wise addition of a sample from ν_λ^m and an independent sample from ν_λ^l.

The SIP(m) process is self-dual [11] with duality functions given by $D(\xi, \eta) = \prod_x d(\xi_x, \eta_x)$, with

$$d(k,l) = \frac{l!}{(l-k)!} \frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+k)} \tag{4.3}$$

where $k \le l$. Self-duality means that

$$\mathbb{E}_\eta^{\mathrm{SIP}(m)} D(\xi, \eta_t) = \mathbb{E}_\xi^{\mathrm{SIP}(m)} D(\xi_t, \eta) \tag{4.4}$$

where $\mathbb{E}_\eta^{\mathrm{SIP}(m)}$ denotes expectation in the SIP process started from the configuration η.

The relation between the polynomials D and the measure ν_λ^m reads

$$\int D(\xi, \eta) \nu_\lambda^m(d\eta) = \left(\frac{\lambda}{1-\lambda}\right)^{|\xi|} \tag{4.5}$$

as follows from a simple computation using the definition of the Γ-function, $\Gamma(r) = \int_0^\infty x^{r-1} e^{-x} dx$.

From conservation of particles in the dual process, we see that self-duality and the relation (4.5) gives stationarity of the measure ν_Λ.

The relation (4.5) can be generalized to "local stationary measure", i.e. the product measures that are obtained from the stationary measure (4.1) by allowing a site-dependent parameter. More precisely, given

$$\overline{\lambda} : S \to [0, 1)$$

we define the local stationary measure associated to the profile $\overline{\lambda}$ by

$$\nu_{\overline{\lambda}} = \bigotimes_{x \in S} \nu_{\overline{\lambda}(x)}^m (d\eta_x) \tag{4.6}$$

For $x_1, \ldots, x_n \in S$ we denote by $\sum_{i=1}^n \delta_{x_i}$ the particle configuration $\xi \in \mathbb{N}^S$ obtained by putting a particles at locations x_i, i.e., $\xi(x) = \sum_{i=1}^n I(x_i = x)$. We then have the following relation between the duality functions and the local stationary measures

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \nu_{\overline{\lambda}}(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{4.7}$$

where

$$\rho(x_i) = \frac{\lambda(x_i)}{1 - \lambda(x_i)}$$

For a constant profile $\overline{\lambda}(x) = \lambda$, $\forall x \in S$, we recover (4.5).

By Lemma 1 below, in the case $S = \mathbb{Z}^d$ and translation invariant finite-range $p(x, y)$, any number of dual particles in the SIP(m) will eventually diffuse away to infinity. From that it is easy to deduce that the measures ν_λ are extremal invariant. To see this, we denote for two finite particle configurations $\xi \perp \xi'$, if their supports are disjoint, i.e., there are no site $x \in S$ where there are ξ and ξ' particles. If $\xi \perp \xi'$ then $D(\xi + \xi', \eta) = D(\xi, \eta) D(\xi', \eta)$. Since at large $t > 0$, assumption (A2) implies that, in the SIP started with a finite number of particles, particles are with probability close to one at different locations (see Lemma 1 for a proof of this), we have that for ξ' a fixed configuration, the event $\xi_t \perp \xi'$ has probability close to one as $t \to \infty$. Therefore

$$\begin{aligned}
&\lim_{t\to\infty} \int \mathbb{E}_\eta^{\mathrm{SIP}(m)}(D(\xi,\eta_t))D(\xi',\eta)\nu_\lambda(d\eta) \\
&= \lim_{t\to\infty} \mathbb{E}_\xi^{\mathrm{SIP}(m)} \int D(\xi_t,\eta)D(\xi',\eta)\nu_\lambda(d\eta) \\
&= \lim_{t\to\infty} \mathbb{E}_\xi^{\mathrm{SIP}(m)} \int D(\xi_t,\eta)D(\xi',\eta)I(\xi_t \perp \xi')\nu_\lambda(d\eta) \\
&= \lim_{t\to\infty} \rho_\lambda^{|\xi_t|+|\xi_t'|} \\
&= \rho_\lambda^{|\xi|+|\xi'|} \\
&= \int D(\xi,\eta)\nu_\lambda(d\eta) \int D(\xi',\eta)\nu_\lambda(d\eta)
\end{aligned} \tag{4.8}$$

which shows that time-dependent correlations of (linear combinations of) $D(\xi,\cdot)$ polynomials decay in the course of time to zero, and hence, by standard arguments, ν_λ is mixing and thus ergodic.

5 Correlation Inequalities in the SIP(m)

For a probability measure μ on the configuration space $\mathbb{N}^S$, we denote its "duality moment function" $K_\mu : S^n \to \mathbb{R}$ by

$$K_\mu(x_1,\ldots,x_n) = \int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right)\mu(d\eta) \tag{5.1}$$

If $\mu = \nu_{\overline{\lambda}}$ is a local stationary measure with profile $\overline{\lambda}$, then

$$K_{\nu_{\overline{\lambda}}}(x_1,\ldots,x_n) = \prod_{i=1}^n \rho(x_i) \tag{5.2}$$

which is clearly positive definite and symmetric. We can therefore apply Theorem 1 and obtain the following result.

Proposition 1 *For all $t \geq 0$, for all profiles $\overline{\lambda} : S \to [0,1)$ and for all $x_1,\ldots,x_n \in S$ we have*

$$K_{\nu_{\overline{\lambda}}S_t}(x_1,\ldots,x_n) \geq \prod_{i=1}^n K_{\nu_{\overline{\lambda}}S_t}(x_i) \tag{5.4}$$

where S_t denotes the semigroup of the SIP(m) *process. In particular, when the* SIP(m) *is started from $\nu_{\overline{\lambda}}$, the random variables $\{\eta_t(x), x \in S\}$ are positively correlated, i.e., for $(x,y) \in S \times S$*

$$\int \mathbb{E}_\eta^{\mathrm{SIP}(m)}(\eta_t(x)\eta_t(y))\nu_{\overline{\lambda}}(d\eta) \geq \int \mathbb{E}_\eta^{\mathrm{SIP}(m)}(\eta_t(x))\nu_{\overline{\lambda}}(d\eta) \int \mathbb{E}_\eta^{\mathrm{SIP}(m)}(\eta_t(y))\nu_{\overline{\lambda}}(d\eta)$$

Proof Denote by $\mathbb{E}^{\mathrm{SIP}(m)}_{x_1,\dots,x_n}$ expectation in the SIP(m) process started with n particles at positions $(x_1,\dots,x_n)$, by $\mathbb{E}^{\mathrm{IRW}(m)}$ expectation in the process of independent random walkers (jumping at rate $2m$) and $\mathbb{E}^{\mathrm{RW}(m)}$ a single random walker expectation. We then have the following chain of inequalities, which is obtained by using sequentially the following: self-duality property (4.4), the comparison inequality (3.8), the relation between the measure $\nu_{\overline{\lambda}}$ and the duality function D (4.7), the independence between random walkers, the fact that a single SIP particle moves as a continuous time random walk, and finally again self-duality (4.4)

$$
\begin{aligned}
&\int \mathbb{E}^{\mathrm{SIP}(m)}_{\eta} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right)\nu_{\overline{\lambda}}(d\eta)\\
&= \mathbb{E}^{\mathrm{SIP}(m)}_{x_1,\dots,x_n} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right)\nu_{\overline{\lambda}}(d\eta)\\
&\geq \mathbb{E}^{\mathrm{IRW}(m)}_{x_1,\dots,x_n} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right)\nu_{\overline{\lambda}}(d\eta)\\
&= \mathbb{E}^{\mathrm{IRW}(m)}_{x_1,\dots,x_n}\left(\prod_{i=1}^n \rho(X_i(t))\right)\\
&= \prod_{i=1}^n \mathbb{E}^{\mathrm{RW}(m)}_{x_i}\rho(X_i(t))\\
&= \prod_{i=1}^n \int \mathbb{E}^{\mathrm{SIP}(m)}_{x_i}(D(\delta_{X_i(t)}, \eta))\nu_{\overline{\lambda}}(d\eta)\\
&= \prod_{i=1}^n \int \mathbb{E}^{\mathrm{SIP}(m)}_{\eta}(D(\delta_{x_i}, \eta_t))\nu_{\overline{\lambda}}(d\eta)
\end{aligned}
\tag{5.5}
$$

□

This proposition shows that starting from a local stationary measure $\nu_{\overline{\lambda}}$, the density profile $\rho_t(x) = \mathbb{E}^{\mathrm{RW}(m)}_x \rho_t(x)$ predicts (by duality) correctly the density at time $t > 0$ but the true measure at time $t > 0$, $\nu_{\overline{\lambda}} S_t$, lies above (in the sense of expectations of D-functions) the product measure with density profile $\rho_t(x)$.

From the analogy with the SEP emphasized above, one could think that (5.4) extends to the case when the SIP process is started from a general product measure. However, for general probability measures μ on Ω, the duality moment function $K_\mu : S^n \to R$ defined in (5.1) is not necessarily positive definite (as is the case for the special product measures $\nu_{\overline{\lambda}}$), since we do not have the equality $D(\sum_{i=1}^n \delta_{x_i}, \eta) = \prod_{i=1}^n D(\delta_{x_i}, \eta)$ in general. Notice that this problem does not appear in the context of the standard SEP, as for that model, the self-duality functions are

$$
D_{\mathrm{SEP}}\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) = \prod_{i=1}^n \eta_{x_i} = \prod_{i=1}^n D_{\mathrm{SEP}}(\delta_{x_i}, \eta)
$$

and hence automatically, for any measure μ, the function K_μ is positive definite in that model.

If however *all* x_i *are different*, we have $D(\sum_{i=1}^n \delta_{x_i}, \eta) = \prod_{i=1}^n D(\delta_{x_i}, \eta)$. For every probability measure μ on Ω, the function $\Psi_\mu : S^n \to \mathbb{R}$ defined by

$$\Psi_\mu(x_1, \ldots, x_n) = \int \prod_{i=1}^n D(\delta_{x_i}, \eta)\mu(d\eta) \tag{5.6}$$

is clearly positive definite. This, together with the fact that under assumption (A2), a finite number of SIP(m) particles diffuse and therefore eventually will be typically at different positions, suggests that in a stationary measure, the variables η_{x_i} are positively correlated.

To state this result we introduce the class of probability measures with uniform finite moments

$$\mathcal{P}_f =: \left\{ \mu : \forall n \in \mathbb{N},\ \sup_{|\xi|=n} \int D(\xi, \eta)\mu(d\eta) =: M_\mu^n < \infty \right\} \tag{5.7}$$

For a sequence of measures $\mu_n \in \mathcal{P}_f$, and $\mu \in \mathcal{P}_f$, we define that $\mu_n \to \mu$ if for all ξ finite particle configuration,

$$\lim_{n\to\infty} \int D(\xi, \eta)\mu_n(d\eta) = \int D(\xi, \eta)\mu(d\eta)$$

We can then formulate our next result.

Proposition 2 *Assume* (A1) *and* (A2). *Let* $\nu \in \mathcal{P}_f$ *be a product measure. Let* $S(t)$ *denote the semigroup of the* SIP(m). *Suppose that*

$$\mu = \lim_{n\to\infty} \nu S(t_n) \tag{5.9}$$

for a subsequence $t_n \uparrow \infty$. *Then we have* $\mu \in \mathcal{P}_f$, μ *is invariant and*

$$K_\mu(x_1, \ldots, x_n) \geq \prod_{i=1}^n K_\mu(x_i) \tag{5.10}$$

Proof First, by duality we have, referring to the definition of $\mathcal{P}_f$, for all $t > 0$,

$$\int \mathbb{E}_\eta^{\mathrm{SIP}(m)} D(\xi, \eta_t)\nu(d\eta) = \mathbb{E}_\xi^{\mathrm{SIP}(m)} \int D(\xi_t, \eta)\nu(d\eta) \leq M_\nu^{|\xi|} < \infty$$

which shows that both $\nu S(t_n)$ and μ are elements of $\mathcal{P}_f$. The invariance of μ follows from duality, $\nu \in \mathcal{P}_f$ and Lemma 1.26 in [16], Chap. V.

To proceed with the proof of the proposition, we start with the following lemma, which ensures that, under condition (A2), any number of SIP(m) particles will eventually be at different locations.

Lemma 1 *Assume* (A2). *Start the finite* SIP(m) *with particles at locations* $\{x_1, \ldots, x_n\}$, *then*

$$\lim_{t\to\infty} \mathbb{P}_{x_1,\ldots,x_n}^{\mathrm{SIP}(m)}(\exists i \neq j : X_i(t) = X_j(t)) = 0 \tag{5.12}$$

Proof We give the proof for $m = 1$. The general case is a straightforward extension. Put $\eta := \sum_{i=1}^n \delta_{x_i}$. Using self-duality we can write

$$\begin{aligned}
\mathbb{P}_\eta^{\mathrm{SIP}(1)}(\exists i \neq j : X_i(t) = X_j(t)) &\le \sum_z \mathbb{P}_\eta^{\mathrm{SIP}(1)}(\eta_t^2(z) - \eta_t(z) > 1) \\
&\le \sum_z \mathbb{E}_\eta^{\mathrm{SIP}(1)}(\eta_t^2(z) - \eta_t(z)) \\
&= \frac{3}{4} \sum_z \mathbb{E}_\eta^{\mathrm{SIP}(1)}(D(2\delta_z, \eta_t)) \\
&= \frac{3}{4} \sum_z \mathbb{E}_{z,z}^{\mathrm{SIP}(1)}(D(\delta_{X_t} + \delta_{Y_t}, \eta)) \\
&\le 3 \sum_z \mathbb{E}_{z,z}^{\mathrm{SIP}(1)}(\eta(X_t)\eta(Y_t)) \\
&= 3 \sum_z \sum_{i,j=1}^n \mathbb{E}_{z,z}^{\mathrm{SIP}(1)}(I(X_t = x_i) I(Y_t = x_j)) \\
&\le 3n^2 \sup_{x,y} \mathbb{P}_{x,y}^{\mathrm{SIP}(m)}(X_t = Y_t) \qquad (5.13)
\end{aligned}$$

where in the last step we used the symmetry of the transition probabilities of the SIP(1) (with two particles). □

We now proceed with the proof of the proposition. For $x_1, \ldots, x_n \in S$ we define

$$\left| D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) - \prod_{i=1}^n D(\delta_{x_i}, \eta) \right| = \Delta(x_1, \ldots, x_n, \eta) \qquad (5.14)$$

We have that $\Delta(x_1, \ldots, x_n, \eta) = 0$ if all x_i are different, i.e., if $|\{x_1, \ldots, x_n\}| = n$. Since by assumption (A2) and Lemma 1, the probability that two SIP(m) walkers out of a finite number n of them occupy the same position, i.e. $X_i(t) = X_j(t)$ for some $i \neq j$, vanishes in the limit $t \to \infty$, we conclude, using $\nu \in \mathcal{P}_f$, for any $x_1, \ldots, x_n \in S$,

$$\lim_{t \to \infty} \int \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{SIP}(m)} \Delta(X_1(t), \ldots, X_n(t), \eta) \nu(d\eta) = 0 \qquad (5.15)$$

Moreover from the comparison inequality (3.8) we have, using the notation (5.6)

$$\begin{aligned}
\mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{SIP}(m)} \Psi_\nu(X_1(t), \ldots, X_n(t)) &\ge \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{IRW}(m)} \Psi_\nu(X_1(t), \ldots, X_n(t)) \\
&= \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{IRW}(m)} \int \prod_{i=1}^n D(\delta_{X_i(t)}, \eta) \nu(d\eta) \\
&= \prod_{i=1}^n \mathbb{E}_{x_i}^{\mathrm{RW}(m)} \int D(\delta_{X_i(t)}, \eta) \nu(d\eta) + \epsilon(t) \qquad (5.16)
\end{aligned}$$

where $\epsilon(t) \to 0$ as $t \to \infty$ by assumption (A1), i.e., for large $t > 0$, independent random walkers are at different locations with probability close to one. Therefore, using the definition (5.9), the self-duality property (4.4), the equation (5.15), the equation (5.16), and taking limits along the subsequence t_n we have

$$\begin{aligned}
K_\mu(x_1, \ldots, x_n) &= \lim_{t\to\infty} \int \mathbb{E}_\eta^{\mathrm{SIP}(m)} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \nu(d\eta) \\
&= \lim_{t\to\infty} \int \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{SIP}(m)} D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \nu(d\eta) \\
&= \lim_{t\to\infty} \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{SIP}(m)} \Psi_\nu(X_1(t), \ldots, X_n(t)) \\
&\geq \lim_{t\to\infty} \prod_{i=1}^n \mathbb{E}_{x_i}^{\mathrm{RW}(m)} \int D(\delta_{X_i(t)}, \eta)\nu(d\eta) \\
&= \prod_{i=1}^n K_\mu(x_i) \qquad (5.17)
\end{aligned}$$

□

6 Correlation Inequalities in the SEP(n)

We now consider the application of the generalized Liggett inequality for negative b. The SEP(n) is the Markov process on $\Omega = \{0, 1, \ldots, n\}^S$ with generator

$$Lf(\eta) = \sum_{x,y\in S} \eta(x)(n - \eta(y))p(x, y)(f(\eta^{xy}) - f(\eta)) \qquad (6.1)$$

The stationary measures of this process are products of binomial distributions, i.e., for $\rho \in [0, 1]$,

$$\nu_\rho = \bigotimes_{x\in S} \mathrm{Bin}(n, \rho) \qquad (6.2)$$

Similar to the case of the inclusion process, for a profile $\overline{\rho} : S \to [0, 1]$ we define the local stationary measure

$$\nu_{\overline{\rho}} = \bigotimes_{x\in S} \mathrm{Bin}(n, \overline{\rho}(x))$$

The duality functions for self-duality are given by (see [11])

$$D(\xi, \eta) = \prod_x d(\xi_x, \eta_x) \qquad (6.3)$$

for $\xi \in \Omega$ a configuration with finitely many particles (at most n per site) and with

$$d(k, l) = \frac{\binom{l}{k}}{\binom{n}{k}} \qquad (6.4)$$

The relation between the duality functions and the local stationary measures is, as usual, i.e., for $\xi = \sum_{i=1}^{n} \delta_{x_i} \in \Omega$ (i.e., at most n particles per site), and $\overline{\rho}$ a profile:

$$\int D(\xi, \eta)\nu_{\overline{\rho}}(d\eta) = \prod_{i=1}^{n} \rho(x_i) \tag{6.5}$$

We define, for a probability measure μ on Ω, its duality moment function

$$K_{\mu}(x_1, \ldots, x_n) = \int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right)\mu(d\eta) \tag{6.6}$$

The following proposition is then the analogue of Proposition 1 in this context (with inequality in the other direction since $b < 0$).

Proposition 3 *For $\overline{\rho} : S \to [0, 1]$ a density profile and $t > 0$,*

$$K_{\nu_{\overline{\rho}} S_t}(x_1, \ldots, x_n) \le \prod_{i=1}^{n} K_{\nu_{\overline{\rho}} S_t}(x_i) \tag{6.8}$$

In particular, for starting from $\nu_{\overline{\rho}}$, the variables $\{\eta_t(x) : x \in S\}$ are negatively correlated.

7 Correlation Inequalities for Some Interacting Diffusions

7.1 The Brownian Momentum Process

The Brownian momentum process is a system of interacting diffusions, initially introduced as a model of heat conduction in [9], and analyzed via duality in [10]. It is defined as a Markov process on $X = \mathbb{R}^S$ via the formal generator on local functions:

$$L_{\mathrm{BMP}} f(\eta) = \left(\sum_{x,y \in S} p(x, y)\left(\eta_x \frac{\partial}{\partial \eta_y} - \eta_x \frac{\partial}{\partial \eta_y}\right)^2\right) f(\eta) \tag{7.1}$$

The variable η_x has to be thought of as momentum of an "oscillator" associated to the site $x \in S$. The local kinetic energy η_x^2 has to be thought of as the analogue of the number of particles at site x in the SIP(m) with $m = 1$. The expectation of η_x^2 is interpreted as the local temperature at x.

Defining the polynomials

$$D(n, z) = \frac{z^{2n}}{(2n-1)!!}$$

we have the duality function $D(\xi, \cdot)$ defined on X and indexed by finite particle configurations $\xi \in \mathbb{N}^S$, $\sum_x \xi_x < \infty$:

$$D(\xi, \eta) = \prod_{x \in S} D(\xi_x, \eta_x) \tag{7.2}$$

In [10, 11], we proved the duality relation

$$\mathbb{E}_{\eta}^{\mathrm{BMP}}(D(\xi, \eta_t)) = \mathbb{E}_{\xi}^{\mathrm{SIP}(1)}(D(\xi_t, \eta)) \tag{7.3}$$

As before, for $x_1, \ldots, x_n \in S$ we denote by $\sum_{i=1}^n \delta_{x_i}$ the particle configuration obtained by putting a particle at each x_i.

Let μ be a product of Gaussian measures on X, with site-dependent variance, i.e., for a function $\rho : S \to [0, \infty)$, we define

$$\mu_\rho = \bigotimes_{x \in S} \nu_{\rho(x)}(d\eta_x) \tag{7.4}$$

where

$$\nu_{\rho(x)}(d\eta_x) = \frac{e^{-\eta_x^2/2\rho(x)}}{\sqrt{2\pi\rho(x)}} d\eta_x$$

is the Gaussian measure on $\mathbb{R}$ with mean zero and variance $\rho(x)$. Then we have

$$\int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_\rho(d\eta) = \prod_{i=1}^n \rho(x_i) \tag{7.5}$$

From this expression, it is obvious that the map

$$S^n \to \mathbb{R} : (x_1, \ldots, x_n) \mapsto \int D\left(\sum_{i=1}^n \delta_{x_i}, \eta\right) \mu_\rho(d\eta) \tag{7.6}$$

is positive definite. Therefore, combining the duality property between BMP process and SIP(1) process, (7.3), with Theorem 1 we have the inequality

$$\begin{aligned}
&\int \mathbb{E}_\eta^{\mathrm{BMP}} D\left(\sum_{i=1}^n \delta_{x_i}, \eta_t\right) \mu_\rho(d\eta) \\
&= \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{SIP}(1)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \mu_\rho(d\eta) \\
&\geq \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{IRW}(m)} \int D\left(\sum_{i=1}^n \delta_{X_i(t)}, \eta\right) \mu_\rho(d\eta) \\
&= \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{IRW}(m)} \left(\prod_{i=1}^n \int D(\delta_{X_i(t)}, \eta) \mu_\rho(d\eta)\right) \\
&= \mathbb{E}_{x_1,\ldots,x_n}^{\mathrm{IRW}(m)} \left(\prod_{i=1}^n \rho(X_i(t))\right) \\
&= \prod_{i=1}^n \mathbb{E}_{x_i}^{\mathrm{RW}(m)} \rho(X_i(t)) \\
&= \prod_{i=1}^n \int \mathbb{E}_{x_i}^{\mathrm{SIP}(1)} (D(\delta_{X_i(t)}, \eta)) \mu_\rho(d\eta) \\
&= \prod_{i=1}^n \int \mathbb{E}_\eta^{\mathrm{BMP}} (D(\delta_{x_i}, \eta_t)) \mu_\rho(d\eta)
\end{aligned} \tag{7.7}$$

which is the analogue of Proposition 1 for the BMP process.

In words, it means that the "non-equilibrium temperature profile" is above the temperature profile predicted from the discrete diffusion equation. It also implies that the variables $\{\eta_x^2 : x \in S\}$ are positively correlated under the measure $(\mu_\rho)_t$ for all choices of $\rho, t > 0$.

More precisely, if we denote

$$\rho_t(x) = \mathbb{E}_x^{\mathrm{RW}(m)} \rho(X_t)$$

then we have that η_x^2 at time t has expectation $\rho_t(x)$ when the starting measure is μ_ρ (since a single particle in the SIP(1) moves as a continuous time random walk). The correlation inequality for the BMP which we just derived shows that the true measure at time $t > 0$ when started from a product of Gaussian measures lies stochastically above the Gaussian product measure with mean zero and variance $\rho_t(x)$.

Similarly, we obtain an analogous correlation inequality for the BMP for a measure obtained as a limit of product measures. We define

$$\mathcal{P}_f(X) = \left\{ \mu : \forall n \in \mathbb{N} : \sup_{|\xi|=n} \int D(\xi, \eta)\mu(d\eta) < \infty \right\}$$

Proposition 4 *Assume* (A1) *and* (A2). *Suppose $\nu \in \mathcal{P}_f(X)$ is a product measure and μ is a limit point of the set $\{\nu S(t) : t \geq 0\}$, where $S(t)$ denotes the semigroup of the BMP process. Then we have the inequality*

$$K_\mu(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} K_\mu(x_i)$$

7.2 The Brownian Energy Process

The Brownian energy process with parameter $m > 0$ (notation BEP(m)) is introduced in [11] as the process on state space $X = [0, \infty)^S$, with generator

$$L = \sum_{x,y \in S} p(x, y) L_{xy}^m \tag{7.9}$$

with

$$L_{xy}^m f(\eta) = 4\eta_x \eta_y \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_x} \right)^2 f(\eta) - 2m(\eta_x - \eta_y) \left(\frac{\partial}{\partial \eta_x} - \frac{\partial}{\partial \eta_x} \right) f(\eta) \tag{7.10}$$

This process is dual to the SIP(m) in the following sense. Define, for $\xi \in \mathbb{N}^S$ a finite particle configuration, and $\eta \in X$ the polynomials

$$D(\xi, \eta) = \prod_{x \in S} d(\xi_x, \eta_x) \tag{7.11}$$

with, for $k \in \mathbb{N}$, $y \in [0, \infty)$

$$d(k, y) = y^k \frac{\Gamma(\frac{m}{2})}{2^k \Gamma(\frac{m}{2} + k)} \tag{7.12}$$

then we have

$$\mathbb{E}_{\eta}^{\mathrm{BEP}(m)} D(\xi, \eta_t) = \mathbb{E}_{\xi}^{\mathrm{SIP}(m)} D(\xi_t, \eta) \tag{7.13}$$

As a consequence, extremal invariant measure of the BEP(m) are products of Γ-distributions with shape parameters $m/2$ and scale parameter $\theta > 0$:

$$\nu_\theta(d\eta) = \bigotimes_{x\in S} \nu_\theta(d\eta_x) \tag{7.14}$$

with

$$\nu_\theta(dz) = \frac{1}{\theta^{m/2}\Gamma(\frac{m}{2})} z^{\frac{m}{2}-1} e^{-z/\theta} \tag{7.15}$$

Similarly we define the local stationary measures

$$\nu_{\overline{\theta}} = \bigotimes_{x\in S} \nu_{\overline{\theta}(x)}(d\eta_x) \tag{7.16}$$

with $\overline{\theta} : S \to [0, \infty)$, and the duality moment function of a probability measure μ on X:

$$K_\mu(x_1, \ldots, x_n) = \int D\left(\sum_{i=1}^{n} \delta_{x_i}, \eta\right) \mu(d\eta) \tag{7.17}$$

As a consequence of the correlation inequalities derived for the SIP(m), we derive the following.

Proposition 5

1. *For all* $\overline{\theta} : S \to [0, \infty)$, $t > 0$, *and* $x_1, \ldots, x_n \in S$ *we have*

$$K_{\nu_{\overline{\theta}} S_t^{\mathrm{BEP}(m)}}(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} K_{\nu_{\overline{\theta}} S_t^{\mathrm{BEP}(m)}} \mu(x_i) \tag{7.19}$$

2. *If for some product measure* ν *on* X *with finite moments*, *and a sequence of* $t_n \uparrow \infty$ *the limit*

$$\mu = \lim_{n\to\infty} \nu S^{\mathrm{BEP}(m)}(t_n)$$

exists, *then*

$$K_\mu(x_1, \ldots, x_n) \geq \prod_{i=1}^{n} K_\mu(x_i) \tag{7.20}$$

8 The Boundary Driven SIP(m)

In this section we consider the non-equilibrium one-dimensional model that is obtained by considering particle reservoirs attached to the first and last sites of the chain. We will show that, if one requires reversibility w.r.t. the measure ν_λ^m *and* duality with absorbing boundaries, this uniquely fixes the birth and death rates at the boundaries.

8.1 Duality for the Boundary Driven SIP(m)

The generator of the boundary driven SIP(m) on a chain $\{1, \dots, N\}$ driven at the end points, reads

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_N + \mathcal{L}_{bulk} \tag{8.1}$$

where $\mathcal{L}_{bulk}$ denotes the SIP(m) generator, with nearest neighbor random walk as underlying kernel, i.e.,

$$\begin{aligned}\mathcal{L}_{bulk} f(\eta) = \sum_{x\in\{1,\dots,N-1\}} & 2\eta_x(m+2\eta_{x+1})(f(\eta^{x,x+1}) - f(\eta)) \\ & + 2\eta_{x+1}(m+2\eta_x)(f(\eta^{x+1,x}) - f(\eta))\end{aligned} \tag{8.2}$$

and where $\mathcal{L}_1, \mathcal{L}_N$ are birth and death processes on the first and N-th variable respectively, i.e.,

$$\mathcal{L}_1 f(\eta) = d_L(\eta_1)(f(\eta - \delta_1) - f(\eta)) + b_L(\eta_1)(f(\eta + \delta_1) - f(\eta))$$

and

$$\mathcal{L}_N f(\eta) = d_R(\eta_N)(f(\eta - \delta_N) - f(\eta)) + b_R(\eta_N)(f(\eta + \delta_N) - f(\eta))$$

These generators model contact with respectively the left and right particle reservoir.

The rates d_L, b_L, d_R, b_R are chosen such that detailed balance is satisfied w.r.t. the measure ν_λ^m, with $\lambda = \lambda_L$ for d_L, b_L, and $\lambda = \lambda_R$ for d_R, b_R. More precisely, this means that these rates satisfy

$$b_\alpha(k)\nu_{\lambda_\alpha}^m(k) = d_\alpha(k+1)\nu_{\lambda_\alpha}^m(k+1) \tag{8.3}$$

for $\alpha \in \{L, R\}$.

To state our duality result, we consider functions $\mathcal{D}(\xi, \eta)$ indexed by particle configurations ξ on $\{0, \dots, N+1\}$ defined by

$$\mathcal{D}(\xi, \eta) = \rho_L^{|\xi_0|} D(\xi_{\{1,\dots,N\}}, \eta)\rho_R^{|\xi_{N+1}|} \tag{8.4}$$

where $\rho_\alpha = \rho_{\lambda_\alpha} = \lambda_\alpha/(1-\lambda_\alpha)$, and where we remember that

$$D(k, n) = \frac{n!}{(n-k)!}\frac{\Gamma(\frac{m}{2})}{\Gamma(\frac{m}{2}+k)}$$

is the duality function for the SIP(m). I.e., for the "normal" sites $\{1, \dots, N\}$ we simply have the old duality functions, and for the "extra added" sites $\{0, N+1\}$ we have the expectation of the duality function over the measure ν_λ^m.

We now want duality to hold with duality functions $\mathcal{D}$, and with a dual process that behaves in the bulk as the SIP(m), and which has *absorbing boundaries* at $\{0, N+1\}$. More precisely, we want the generator of the dual process to be

$$\hat{\mathcal{L}} = \mathcal{L}_{bulk} + \hat{\mathcal{L}}_1 + \hat{\mathcal{L}}_N \tag{8.5}$$

with $\mathcal{L}_{bulk}$ given by (8.2), and

$$\hat{\mathcal{L}}_1 f(\xi) = \xi_1 (f(\xi^{1,0}) - f(\xi))$$
$$\hat{\mathcal{L}}_N f(\xi) = \xi_N (f(\xi^{N,N+1}) - f(\xi))$$

for $\xi \in \mathbb{N}^{\{0,1\dots,N+1\}}$. The duality relation then reads, as usual,

$$(\mathcal{L}\mathcal{D}(\xi,\cdot))(\eta) = (\hat{\mathcal{L}}\mathcal{D}(\cdot,\eta))(\xi) \tag{8.6}$$

Since self-duality is satisfied for the bulk generator with the choice (8.4), i.e., since

$$(\mathcal{L}_{bulk}\mathcal{D}(\xi,\cdot))(\eta) = (\mathcal{L}_{bulk}\mathcal{D}(\cdot,\eta))(\xi)$$

(8.6) will be satisfied if we have the following relations at the boundaries: for all $k \le n$:

$$\begin{aligned} &b_\alpha(n)(D(k,n+1) - D(k,n)) + d_\alpha(n)(D(k,n-1) - D(k,n)) \\ &\quad = k(D(k-1,n)\rho_\alpha - D(k,n)) \end{aligned} \tag{8.7}$$

where $\alpha \in \{L, R\}$.

From detailed balance (8.3) we obtain

$$d_\alpha(n) = \frac{1}{\lambda_\alpha}\left(\frac{n}{\frac{m}{2}+n-1}\right) b_\alpha(n-1) \tag{8.8}$$

Working out (8.7) gives, using (4.3),

$$\begin{aligned} &b_\alpha(n)\left(\frac{n+1}{n+1-k} - 1\right) + d_\alpha(n)\left(\frac{n-k}{n} - 1\right) \\ &\quad = k\left(\frac{(\frac{m}{2}+k-1)\rho_\alpha}{n-k+1} - 1\right) \end{aligned} \tag{8.9}$$

which simplifies to

$$\frac{b_\alpha(n)}{n+1-k} - \frac{d_\alpha(n)}{n} = \left(\frac{(\frac{m}{2}+k-1)\rho_\alpha}{n-k+1} - 1\right) \tag{8.10}$$

Choosing

$$d_\alpha(n) = \frac{n}{1-\lambda_\alpha} \tag{8.11}$$

and by the detailed balance condition (8.8),

$$b_\alpha(n) = \left(\frac{m}{2}+n\right)\frac{\lambda_\alpha}{1-\lambda_\alpha} \tag{8.12}$$

it is then an easy computation to see that (8.7) is satisfied with the choices (8.11), (8.12). Indeed, (8.10) reduces to the simple identity

$$\left(\frac{m}{2}+n\right)\left(\frac{\lambda}{1-\lambda}\right)\frac{1}{n+1-k} - \frac{1}{1-\lambda} = \frac{\frac{m}{2}+k-1}{n+1-k}\left(\frac{\lambda}{1-\lambda}\right) - 1$$

We remark that the requirement of detailed balance alone is not sufficient to fix the rates uniquely. However, the additional duality constraint (8.7) does fix the rates to the unique expression given by (8.11) and (8.12).

As a consequence of duality with duality functions (8.4), we have that the boundary driven SIP(m) with generator (8.1) has a unique stationary measure $\mu_{L,R}$ for which expectations of the polynomials $D(\xi,\eta)$ are given in terms of absorption probabilities:

$$\begin{aligned}\int D(\xi,\eta)\mu_{L,R}(d\eta) &= \lim_{t\to\infty}\mathbb{E}_\eta \mathcal{D}(\xi,\eta_t)\\ &= \lim_{t\to\infty}\hat{\mathbb{E}}_\xi \mathcal{D}(\xi_t,\eta)\\ &= \sum_{k,l:k+l=|\xi|}\rho_L^k\rho_R^l\hat{\mathbb{P}}_\xi(\xi_\infty = k\delta_0 + l\delta_{N+1}) \qquad (8.13)\end{aligned}$$

Here, $\hat{\mathbb{E}}_\xi$ denotes expectation in the dual process (which is absorbing at $\{0, N+1\}$) starting from ξ. In particular, since a single SIP(m) particle performs continuous time simple random walk (at rate $2m$) we have a linear density profile, i.e.,

$$\int D(\delta_i,\eta)\mu_{L,R}(d\eta) = \rho_L\left(1-\frac{i}{N+1}\right) + \rho_R\frac{i}{N+1} \qquad (8.14)$$

8.2 Correlation Inequality for the Boundary Driven SIP(m)

For $x_1,\dots,x_n \in \{1,\dots,N\}$ let us denote by $(X_1(t),\dots,X_n(t))$ the positions of particles at time t evolving according to the SIP(m) with absorbing boundary sites at $\{0, N+1\}$, i.e., according to the generator (8.5), and initially at positions $x_1,\dots,x_n$. Let $(Y_1(t),\dots,Y_n(t))$ denote the positions at time t of independent random walkers (jumping at rate $2m$) absorbed (at rate 1) at $\{0, N+1\}$, initially at positions $x_1,\dots,x_n$. Since the absorption parts of the generators of $(X_1(t),\dots,X_n(t))$ and $(Y_1(t),\dots,Y_n(t))$ are the same, we have the same inequality for expectations of positive definite functions as in Theorem 1. Therefore, we have the following result on positivity of correlations in the stationary state. This has once more to be compared to the analogous situation of the boundary driven exclusion process, where the stationary covariances of site-occupations are negative.

Proposition 6 *Let $\mu_{L,R}$ denote the unique stationary measure of the process with generator* (8.1). *Let $x_1,\dots,x_n \in \{1,\dots,N\}$, then we have*

$$\int D\left(\sum_{i=1}^n \delta_{x_i},\eta\right)\mu_{L,R}(d\eta) \ge \prod_{i=1}^n \int D(\delta_{x_i},\eta)\mu_{L,R}(d\eta) \qquad (8.16)$$

In particular, $\eta_x, x \in \{1,\dots,N\}$ are positively correlated under the measure $\mu_{L,R}$.

Proof Start from the measure ν_λ^m. Define the map $\{0,\dots,N+1\}^n \to \mathbb{R}$:

$$(x_1,\dots,x_n) \mapsto \int \mathcal{D}\left(\sum_{i=1}^n \delta_{x_i},\eta\right)\nu_\lambda^m(d\eta) = \prod_{i=1}^n \rho(x_i) \qquad (8.17)$$

where $\rho(x) = \frac{\lambda}{1-\lambda}$ for $x \in \{1,\dots,N\}$ and $\rho(0) = \rho_L$, $\rho(N+1) = \rho_R$. This is clearly positive definite. Therefore, for $x_1,\dots,x_n \in \{1,\dots,N\}$, we have

$$
\begin{aligned}
\int D\left(\sum_{i=1}^{n}\delta_{x_i},\eta\right)\mu_{L,R}(d\eta) &= \lim_{t\to\infty}\int \mathbb{E}_\eta \mathcal{D}\left(\sum_{i=1}^{n}\delta_{x_i},\eta_t\right)\nu_\lambda^m(d\eta)\\
&= \lim_{t\to\infty}\int \hat{\mathbb{E}}^{\mathrm{SIP}(m),abs}_{x_1,\dots,x_n}\left(\mathcal{D}\left(\sum_{i=1}^{n}\delta_{X_i(t)},\eta\right)\right)\nu_\lambda^m(d\eta)\\
&\geq \lim_{t\to\infty}\hat{\mathbb{E}}^{\mathrm{IRW}(m),abs}_{x_1,\dots,x_n}\left(\int \mathcal{D}\left(\sum_{i=1}^{n}\delta_{X_i(t)},\eta\right)\nu_\lambda^m(d\eta)\right)\\
&= \prod_{i=1}^{n}\lim_{t\to\infty}\hat{\mathbb{E}}^{\mathrm{IRW}(m),abs}_{x_i}\rho(X_i(t))\\
&= \prod_{i=1}^{n}\int D(\delta_{x_i},\eta)\mu_{L,R}(d\eta) \qquad (8.18)
\end{aligned}
$$

where we denoted $\hat{\mathbb{E}}^{\mathrm{SIP}(m),abs}$ for expectation over SIP(m) particles absorbed at $\{0, N+1\}$, and $\hat{\mathbb{E}}^{\mathrm{IRW}(m),abs}$ for expectation over a system of independent random walkers (jumping at rate $2m$) absorbed (at rate 1) at $\{0, N+1\}$. □

Remark 1

1. Proposition 6 is in agreement with the findings of [10], where the covariance of η_i, η_j in the measure $\mu_{L,R}$ was computed explicitly, and turned out to be positive.
2. For the nearest neighbor SEP on $\{1,\dots,N\}$ driven at the boundaries, we have self-duality with absorption of dual particles at $\{0, N+1\}$ and duality function

$$
\mathcal{D}_{\mathrm{SEP}}\left(\sum_{i=1}^{n}\delta_{x_i},\eta\right) = \prod_{i=1}^{n}\eta_{x_i}
$$

   where $\eta_0 := \rho_L, \eta_{N+1} = \rho_R$. Since for SEP particles we have the comparison inequality of Liggett, we have as an analogue of (8.16) in the SEP context,

$$
\int \prod_{i=1}^{n}\eta_{x_i}\ \mu_{L,R}(d\eta) \leq \prod_{i=1}^{n}\int \eta_{x_i}\ \mu_{L,R}(d\eta)
$$

   i.e., η_{x_i} are negatively correlated. The same holds for the non-equilibrium SEP(n) driven by appropriate boundary generators. This is in agreement with the results in [17], where the two-point function of the measure $\mu_{L,R}$ is computed, and with the work of [7], where some multiple correlations are explicitly computed.
3. We expect the KMP-model, a model of heat conduction introduced and studied in [8] to also have positive correlations. Indeed, the KMP and the BEP(2) model are related by a so-called instantaneous thermalization limit [11]. Therefore, it is natural to think that similar correlation inequalities should hold for the KMP as we have derived for the BEP. The limit to obtain the KMP from the BEP is however difficult to perform on the level of the n-particle representation and it is thus not clear (to us) how to prove that the KMP preserves the positive correlation structure of the BEP. A positive hint in this direction comes from the explicit expression of the two point function which has been computed for the KMP in the non-equilibrium context in [4].

References

1. Andjel, E.: A correlation inequality for the symmetric exclusion process. Ann. Probab. **16**, 717–721 (1988)
2. Bernardin, C.: Superdiffusivity of asymmetric energy model in dimensions 1 and 2. J. Math. Phys. **49**(10), 103301 (2008)
3. Bernardin, C., Olla, S.: Fourier's law for a microscopic model of heat conduction. J. Stat. Phys. **121**, 271–289 (2005)
4. Bertini, L., De Sole, A., Gabrielli, D., Jona-Lasinio, G., Landim, C.: Stochastic interacting particle systems out of equilibrium. J. Stat. Mech., Theory Exp. P07014n (2007)
5. Borcea, J., Brändén, P., Liggett, T.M.: Negative dependence and the geometry of polynomials. J. Am. Math. Soc. **22**, 521–567 (2009)
6. De Masi, A., Presutti, E.: Mathematical Methods for Hydrodynamic Limits. Lecture Notes in Mathematics, vol. 1501. Springer, Berlin (1991)
7. Derrida, B., Lebowitz, J.L., Speer, E.R.: Entropy of open lattice systems. J. Stat. Phys. **126**, 1083–1108 (2007)
8. Galves, A., Kipnis, C., Marchioro, C., Presutti, E.: Nonequilibrium measures which exhibit a temperature gradient: study of a model. Commun. Math. Phys. **81**, 127–147 (1981)
9. Giardina, C., Kurchan, J.: The Fourier law in a momentum-conserving chain. J. Stat. Mech. P05009 (2005)
10. Giardina, C., Kurchan, J., Redig, F.: Duality and exact correlations for a model of heat conduction. J. Math. Phys. **48**, 033301 (2007)
11. Giardina, C., Kurchan, J., Redig, F., Vafayi, K.: Duality and hidden symmetries in interacting particle systems. J. Stat. Phys. **135**, 25–55 (2009)
12. Gobron, T., Saada, E.: Coupling, attractiveness and hydrodynamics for conservative particle systems. Preprint available on arxiv.org (2009)
13. Harris, T.E.: A correlation inequality for Markov processes in partially ordered state spaces. Ann. Probab. **5**, 451–454 (1977)
14. Inglis, J., Neklyudov, M., Zegarlinski, B.: Ergodicity for infinite particle systems with locally conserved quantities. arXiv:1002.0282v2 (2010)
15. Liggett, T.M.: Negative correlations and particle systems. Markov Process. Relat. Fields **8**, 547–564 (2002)
16. Liggett, T.M.: Interacting Particle Systems. Classics in Mathematics. Springer, Berlin (2005). Reprint of the 1985 original
17. Spohn, H.: Long range correlations for stochastic lattice gases in a non-equilibrium steady state. J. Phys. A **16**, 4275–4291 (1983)